*Journal of Statistical Physics, Vol. 122, No. 1, January 2006 (© 2006)*
DOI: 10.1007/s10955-005-8068-9

# New Characterizations of the Region of Complete Localization for Random Schrödinger Operators

**Francois Germinet[1] and Abel Klein[2]**

*Received March 8, 2005; accepted July 5, 2005*

We study the region of complete localization in a class of random operators which includes random Schrödinger operators with Anderson-type potentials and classical wave operators in random media, as well as the Anderson tight-binding model. We establish new characterizations or criteria for this region of complete localization, given either by the decay of eigenfunction correlations or by the decay of Fermi projections. (These are necessary and sufficient conditions for the random operator to exhibit complete localization in this energy region.) Using the first type of characterization we prove that in the region of complete localization the random operator has eigenvalues with finite multiplicity.

## 1. INTRODUCTION

We study localization in a class of random operators which includes random Schrödinger operators with Anderson-type potentials and classical wave operators in random media, as well as the Anderson tight-binding model. For these operators localization is obtained either by a multiscale analysis,(7,10,13–16,22,23,25–30,33,35,36,40,41,43,44,46,48,53,54,56) or, in certain cases, by the fractional moment method.(1,2,4,5,47,57) In addition to pure point spectrum with exponentially localized eigenfunctions, localization proved by a either a multiscale analysis or the fractional moment method always include other properties such as dynamical localization.(1,2,5,17,32,33)

In ref.(37) we proved a converse to the multiscale analysis: the region of dynamical localization coincides with the region where the multiscale analysis

[1] Université de Cergy-Pontoise, Département de Mathématiques, Site de Saint-Martin, 2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise cedex, France; e-mail: germinet@math.u-cergy.fr
[2] University of California, Irvine, Department of Mathematics, Irvine, CA 92697-3875, USA. e-mail: aklein@uci.edu

(and the fractional moment method, when applicable) can be performed. We also gave a large list of characterizations of this region of localization, that is, necessary and sufficient conditions to be satisfied by the random operator in this energy region for a multiscale analysis to be performed at these energies (Ref.[37], *Theorem* 4.2). This region of localization is the analogue for random operators of the region of complete analyticity for classical spin systems[20,21]. For this reason we call it the *region of complete localization*. (Note that the spectral region of complete localization is called the strong insulator region in ref.[37] and the region of complete localization is called the region of dynamical localization in ref.[39].)

In this article we establish two new consequences of the multiscale analysis that are also characterizations of the region of complete localization, given either by the decay of eigenfunction correlations or by the decay of Fermi projections. Using the characterization by the decay of eigenfunction correlations we prove that in the region of complete localization the random operator has eigenvalues with finite multiplicity.

In the one-dimensional case the multiplicity of eigenvalues is easily seen to be always less than or equal to 2. But for $d > 1$ this had only been previously known for in two cases. The first is the Anderson tight-binding model with bounded density for the probability distribution of the single site potential, which has simple eigenvalues in the region of localization[45,52] The second is its continuum analogue, Anderson-type Hamiltonians in the continuum with bounded density for the probability distribution of the strength of single site potential, for which the finite multiplicity of eigenvalues in the region of localization is known[13]. (Although Simon's original proof for the Anderson model[52] does not shed light on the continuum, the recent proof by Klein and Molchanov[45] indicates that these Anderson-type Hamiltonians in the continuum should have simple eigenvalues in the region of localization. The missing ingredient is a continuous analogue of Minami's estimate.[49])

Our proof of finite multiplicity of eigenvalues only requires the conditions for the multiscale analysis, so it applies in great generality. It neither requires probability distributions with bounded densities, nor the unique continuation property for eigenfunctions, both requirements for the Combes and Hislop result[13]. In particular, our result applies to random Landau Hamiltonians[14,35,39,56] and to classical wave operators (e.g., acoustic and Maxwell operators) in random media[27,28,43]

We first characterize the region of complete localization by the decay of the expectation of eigenfunction correlations (Theorem 1). We call this characterization the strong form of "Summable Uniform Decay of Eigenfunction Correlations" (SUDEC). SUDEC has also an almost-sure version which is essentially equivalent to the SULE ("Semi Uniformly Localized Eigenfunctions") property introduced in ref. [18, 19]. This almost-sure SUDEC is a modification of the WULE ("Weakly Uniformly Localized Eigenfunctions") property in ref.[31]. (See also ref.[55] for related properties.) But although SUDEC has a strong form (i.e., in expectation), SULE does not by its very definition.

Recently detailed almost-sure properties of localization like SULE or SUDEC, which go beyond exponential localization or almost-sure dynamical localization, turned out to be crucial in the analysis of the quantum Hall effect. In ref.[24], SULE is used to prove the equivalence between edge and bulk conductance in quantum Hall systems whenever the Fermi energy falls into a region of localized states. In refs. [11, 12], SUDEC is used to regularize the edge conductance in the region of localized states and get its quantization to the desired value. In ref.[39], SUDEC is the main ingredient for a new and quite transparent proof of the constancy of the bulk conductance if the Fermi energy lies in a region of localized states.

It is well known that in the region of complete localization the random operator has pure point spectrum with exponentially decaying eigenfunctions. [29, 23, 41] The SULE property is also known with exponentially decaying eigenfunctions. [32, 33] Theorem 1 yields easily an almost-sure SUDEC (and SULE) with sub-exponentially decaying eigenfunctions. Combining the proof of ref.[31] Theorem[1.5] with the argument in refs. [23, 41], we obtain a form of SUDEC with exponentially decaying eigenfunctions (Theorem 2). (See ref.[38] for more on SUDEC and SULE.)

We conclude with a characterization of the region of complete localization by the decay of the expectation of the operator kernel of Fermi projections (Theorem 3), a crucial ingredient in linear response theory and in explanations of the quantum Hall effect. [3, 6, 9, 39]

The derivation of SUDEC and of the decay of Fermi projections from the multiscale analysis is based on the methods developed in ref.[33] and, in the case of the Fermi projections, the sub-exponential kernel decay for Gevrey-like functions of generalized Schrödinger operators given in ref.[8]. That they characterize the region of complete localization relies on the converse to the multiscale analysis, the fact that slow transport implies that a multiscale analysis can be performed [37].

This article is organized as follows: We introduce random operators, state our assumptions, and define the region of complete localization in Section 2 We state our results in Section 3 Theorem 1 and its corollaries are proved in Section 4 Theorem 2 is proved in Section 5 The proof of Theorem 3 is given in Section 6

**Notation:** We set $\langle x\rangle := \sqrt{1+|x|^2}$ for $x \in \mathbb{R}^d$. By $\Lambda_L(x)$ we denote the open cube (or box) $\Lambda_L(x)$ in $\mathbb{R}^d$ (or $\mathbb{Z}^d$), centered at $x \in \mathbb{Z}^d$ with side of length $L > 0$; we write $\chi_{x,L}$ for its characteristic function, and set $\chi_x := \chi_{x,1}$. Given an open interval $I \subset \mathbb{R}$, we denote by $C_c^\infty(I)$ the class of real valued infinitely differentiable functions on $\mathbb{R}$ with compact support contained in $I$, with $C_{c,+}^\infty(I)$ being the subclass of nonnegative functions. The Hilbert-Schmidt norm of an operator $A$ is written as $\|A\|_2$, i.e., $\|A\|_2^2 = \operatorname{tr} A^*A$. $C_{a,b,\ldots}$, $K_{a,b,\ldots}$, etc., will always denote some finite constant depending only on $a, b, \ldots$. (We omit the dependence on the dimension $d$ in final results.)

## 2. RANDOM OPERATORS AND THE REGION OF COMPLETE LOCALIZATION

In this article *a random operator is a* $\mathbb{Z}^d$*-ergodic measurable map* $H_\omega$ *from a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *(with expectation* $\mathbb{E}$*) to generalized Schrödinger operators on the Hilbert space* $\mathcal{H}$*, where either* $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x; \mathbb{C}^n)$ *or* $\mathcal{H} = \ell^2(\mathbb{Z}^d; \mathbb{C}^n)$. Generalized Schrödinger operators are a class of semibounded second order partial differential operators of Mathematical Physics, which includes the Schrödinger operator, the magnetic Schrödinger operator, and the classical wave operators, eg., the acoustic operator and the Maxwell operator. (See ref.[34] for a precise definition and [41] for examples.) *We assume that* $H_\omega$ *satisfies the standard conditions for a generalized Schrödinger operator with constants uniform in* $\omega$*.*

Measurability of $H_\omega$ means that the mappings $\omega \to f(H_\omega)$ are weakly (and hence strongly) measurable for all bounded Borel measurable functions $f$ on $\mathbb{R}$. $H_\omega$ is $\mathbb{Z}^d$-ergodic if there exists a group representation of $\mathbb{Z}^d$ by an ergodic family $\{\tau_y;\ y \in \mathbb{Z}^d\}$ of measure preserving transformations on $(\Omega, \mathcal{F}, \mathbb{P})$ such that we have the covariance given by

$$U(y) H_\omega U(y)^* = H_{\tau_y(\omega)} \qquad \text{for all } y \in \mathbb{Z}^d, \tag{2.1}$$

where $U(y)$ is the unitary operator given by translation: $(U(y)f)(x) = f(x - y)$. (Note that for Landau Hamiltonians translations are replaced by magnetic translations.) It follows that there exists a nonrandom set $\Sigma$ such that $\sigma(H_\omega) = \Sigma$ with probability one, where $\sigma(A)$ denotes the spectrum of the operator $A$. In addition, the decomposition of $\sigma(H_\omega)$ into pure point spectrum, absolutely continuous spectrum, and singular continuous spectrum is also the same with probability one. (E.g., refs. [50, 54].)

*We assume that the random operator* $H_\omega$ *satisfies the hypotheses of*[33, 37] *in an open energy interval* $\mathcal{I}$. These were called assumptions or properties SGEE, SLI, EDI, IAD, NE, and W in refs. [33, 35, 37, 41]. (Although the results in ref.[37] are written for random Schrödinger operators, they hold without change for generalized Schrödinger operators as long as these hypotheses are satisfied.) Although we assume a polynomial Wegner estimate as in ref.[37], our results are still valid if we only have a sub-exponential Wegner estimate, with the caveat that one must substitute sub-exponential moments for polynomial moments (see ref.[37] Remark 2.3). In particular, our results apply to Anderson or Anderson-type Hamiltonians without the requirement of a bounded density for the probability distribution of the single site potential.

Property SGEE guarantees the existence of a generalized eigenfunction expansion in the strong sense. *We assume that* $H_\omega$ *satisfies the stronger trace estimate* ref.[33] Eq. (2.36), as in ref.[37]. (Note that for classical wave operators we always project to the orthogonal complement of the kernel of $H_\omega$, see refs. [33, 42, 43].) For some fixed $\kappa > \frac{d}{2}$ (which will be generally omitted from the notation) we let $T_a$ denote the operator on $\mathcal{H}$ given by multiplication by the function $\langle x-a\rangle^\kappa$, $a \in \mathbb{Z}^d$, with $T := T_0$. Since $\langle a+b\rangle \le \sqrt{2}\langle a\rangle\langle b\rangle$, we have

$$\|T_b T_a^{-1}\| \le 2^{\frac{\kappa}{2}} \langle b-a\rangle^\kappa. \tag{2.2}$$

The domain of $T$, $\mathcal{D}(T)$, equipped with the norm $\|\phi\|_+ = \|T\phi\|$, is a Hilbert space, denoted by $\mathcal{H}_+$ . The Hilbert space $\mathcal{H}_-$ is defined as the completion of $\mathcal{H}$ in the norm $\|\psi\|_- = \|T^{-1}\psi\|$. By construction, $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ , and the natural injections $\imath_+ : \mathcal{H}_+ \to \mathcal{H}$ and $\imath_- : \mathcal{H} \to \mathcal{H}_-$ are continuous with dense range. The operators $T_+ : \mathcal{H}_+ \to \mathcal{H}$ and $T_- : \mathcal{H} \to \mathcal{H}_-$, defined by $T_+ = T\imath_+$ , and $T_- = \imath_- T$ on $\mathcal{D}(T)$, are unitary. We define the random spectral measure

$$\mu_\omega(B) := \operatorname{tr}\{T^{-1}P_{B,\omega}T^{-1}\} = \|T^{-1}P_{B,\omega}\|_2^2, \tag{2.3}$$

where $B \subset \mathbb{R}$ is a Borel set and $P_{B,\omega} = \chi_B(H_\omega)$. It follows from[33], Eq. (2.36) that for $\mathbb{P}$-a.e. $\omega$ we have

$$\mu_\omega(B) = \mu_\omega(B\cap\Sigma) \le K_{B\cap\Sigma}, \tag{2.4}$$

where $K_B := K_{B\cap\Sigma}$ is independent of $\omega$, increasing in $B\cap\Sigma$, and $K_B < \infty$ if $B\cap\Sigma$ is bounded. Using the covariance (2.1), for $\mathbb{P}$-a.e. $\omega$ and all $a \in \mathbb{Z}^d$ we have

$$\mu_{a,\omega}(B) := \|T_a^{-1}P_{B,\omega}\|_2^2 = \|T^{-1}P_{B,\tau(-a)\omega}\|_2^2 = \mu_{\tau(-a)\omega}(B) \le K_B. \tag{2.5}$$

We have a generalized eigenfunction expansion for $H_\omega$: For $\mathbb{P}$-a.e. $\omega$ there exists a $\mu_\omega$-locally integrable function $\mathbf{P}_\omega(\lambda)\colon \mathbb{R} \to \mathcal{T}_1(\mathcal{H}_+,\mathcal{H}_-)$, the Banach space of bounded operators $A\colon \mathcal{H}_+ \to \mathcal{H}_-$ with $T_-^{-1}AT_+^{-1}$ trace class, such that

$$\operatorname{tr}\left\{T_-^{-1}\mathbf{P}_\omega(\lambda)T_+^{-1}\right\} = 1 \quad \text{for } \mu_\omega\text{-a.e. } \lambda, \tag{2.6}$$

and, for all Borel sets $B$ with $B\cap\Sigma$ bounded,

$$\imath_- P_\omega(B)\imath_+ = \int_B \mathbf{P}_\omega(\lambda)\, d\mu_\omega(\lambda), \tag{2.7}$$

where the integral is the Bochner integral of $\mathcal{T}_1(\mathcal{H}_+,\mathcal{H}_-)$-valued functions. Moreover, if $\phi \in \mathcal{H}_+$, then $\mathbf{P}_\omega(\lambda)\phi \in \mathcal{H}_-$ is a generalized eigenfunction of $H_\omega$ with generalized eigenvalue $\lambda$ (i.e., an eigenfunction of the closure of $H_\omega$ in $\mathcal{H}_-$ with eigenvalue $\lambda$) for $\mu_\omega$-a.e $\lambda$. (See ref.[42], Section 3, for details.)

The multiscale analysis requires the notion of a *finite volume operator*, a "restriction" $H_{\omega,x,L}$ of $H_\omega$ to the cube (or box) $\Lambda_L(x)$, centered at $x \in \mathbb{Z}^d$ with side of length $L \in 2\mathbb{N}$ (assumed here for convenience; we may take $L \in L_0\mathbb{N}$ for a suitable $L_0 1$ as in ref.[39]), where the "randomness based outside the cube $\Lambda_L(x)$" is not taken into account. *We assume the existence of appropriate finite volume* *operators $H_{\omega,x,L}$ for $x \in \mathbb{Z}^d$ with $L \in 2\mathbb{N}$ satisfying properties SLI, EDI, IAD, NE, and W in the open interval $\mathcal{I}$.* (See the discussion in ref.[39], Section 4.)

*The region of complete localization $\Xi_{\mathcal{I}}^{CL}$ for the random operator $H_\omega$ in the open interval I is defined as the set of energies $E \in \mathcal{I}$ where we have the conclusions of the bootstrap multiscale analysis*, i.e., as the set of $E \in \mathcal{I}$ for which there exists some open interval $I \subset \mathcal{I}$, with $E \in I$, such that given any $\zeta$, $0 < \zeta < 1$, and $\alpha$, $1 < \alpha < \zeta^{-1}$, there is a length scale $L_0 \in 6\mathbb{N}$ and a mass $m > 0$, so if we set $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}}$, $k = 0, 1, \ldots,$ we have

$$\mathbb{P}\,\{R\,(m, L_k, I, x, y)\}\, 1 - \mathrm{e}^{-L_k^\zeta} \tag{2.8}$$

for all $k = 0, 1, \ldots$, and $x, y \in \mathbb{Z}^d$ with $|x - y| > L_k + \varrho$, where

$$R(m, L, I, x, y) = \tag{2.9}$$
$$\{\omega;\ \text{for every } E' \in I \text{ either } \Lambda_L(x) \text{ or } \Lambda_L(y) \text{ is } (\omega, m, E')\text{-regular}\}\,.$$

Here $[K]_{6\mathbb{N}} = \max\{L \in 6\mathbb{N};\ L \le K\}$ (we work with scales in $6\mathbb{N}$ for convenience); $\rho > 0$ is given in Assumption IAD, if $\operatorname{dist}(\Lambda_L(x), \Lambda_{L'}(x')) > \varrho$, then events based in $\Lambda_L(x)$ and $\Lambda_{L'}(x')$ are independent. Given $E \in \mathbb{R}$, $x \in \mathbb{Z}^d$ and $L \in 6\mathbb{N}$, we say that the box $\Lambda_L(x)$ is $(\omega, m, E)$-regular for a given $m > 0$ if $E \notin \sigma(H_{\omega,x,L})$ and

$$\|\Gamma_{x,L} R_{\omega,x,L}(E)\chi_{x,\frac{L}{3}}\| \le \mathrm{e}^{-m\frac{L}{2}}, \tag{2.10}$$

where $R_{\omega,x,L}(E) = (H_{\omega,x,L} - E)^{-1}$ and $\Gamma_{x,L}$ denotes the charateristic function of the "belt" $\overline{\Lambda}_{L-1}(x)\backslash\Lambda_{L-3}(x)$. (See refs.[33, 41]). We will take $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x; \mathbb{C}^n)$, but the arguments can be easily modified for $\mathcal{H} = \ell^2(\mathbb{Z}^d; \mathbb{C}^n)$.)

By construction $\Xi_{\mathcal{I}}^{\mathrm{CL}}$ is an open set. It can be defined in many ways, we gave the most convenient definition for our purposes. (We refer to ref.[37], Theorem 4.2) for the equivalent properties that characterize $\Xi_{\mathcal{I}}^{\mathrm{CL}}$. The spectral region of complete localization in $\mathcal{I}$, $\Xi_{\mathcal{I}}^{\mathrm{CL}} \cap \Sigma$, is called the "strong insulator region" in ref.[37].) Note that $\Xi_{\mathcal{I}}^{\mathrm{CL}}$ is the set of energies in $\mathcal{I}$ where we can perform the bootstrap multiscale analysis. (If the conditions for the fractional moment method are satisfied in $\mathcal{I}$, $\Xi_{\mathcal{I}}^{\mathrm{CL}}$ coincides with the set of energies in $\mathcal{I}$ where the fractional moment method can be performed.) By our definition spectral gaps are (trivially) intervals of complete localization.

## 3. THEOREMS AND COROLLARIES

In this article we provide two new characterizations of the region of complete localization. The first characterizes the region of complete localization by the decay of the expectation of generalized eigenfunction correlations, the second by the expectation of decay of Fermi projections.

We start with generalized eigenfunctions. Given $\lambda \in \mathbb{R}$ and $a \in \mathbb{Z}^d$ we set

$$\mathbf{W}_{\lambda,\omega}(a) := \begin{cases} \sup\limits_{\substack{\phi \in \mathcal{H}_+ \\ \mathbf{P}_\omega(\lambda)\phi \neq 0}} \dfrac{\|\chi_a \mathbf{P}_\omega(\lambda)\phi\|}{\|T_a^{-1}\mathbf{P}_\omega(\lambda)\phi\|} & \text{if } \mathbf{P}_\omega(\lambda) \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

$\mathbf{W}_{\lambda,\omega}(a)$ is a measurable function of $(\lambda, \omega)$ for each $a \in \mathbb{Z}^d$ with

$$\mathbf{W}_{\lambda,\omega}(a) \leq \langle \tfrac{\sqrt{d}}{2} \rangle^\kappa = \left(1 + \tfrac{d}{4}\right)^{\frac{\kappa}{2}} . \tag{3.2}$$

Our first characterization is given in the following theorem.

**Theorem 1.** *Let $I$ be a bounded open interval with $\bar{I} \subset \mathcal{I}$. If $\bar{I} \subset \Xi_{\mathcal{I}}^{\mathrm{CL}}$, then for all $\zeta \in ]0,1[$ we have*

$$\mathbb{E}\left\{ \left\| \mathbf{W}_{\lambda,\omega}(x)\mathbf{W}_{\lambda,\omega}(y) \right\|_{\mathrm{L}^\infty(I, \mathrm{d}\mu_\omega(\lambda))} \right\} \leq C_{I,\zeta}\, \mathrm{e}^{-|x-y|^\zeta} \quad \text{for all } x, y \in \mathbb{Z}^d. \tag{3.3}$$

*Conversely, if* (3.3) *holds for some $\zeta \in ]0,1[$, then $I \subset \Xi_{\mathcal{I}}^{\mathrm{CL}}$.*

Note that the converse will still hold if we only have fast enough polynomial decay in (3.3).

**Remark 1.** *We may replace the denominator $\|T_a^{-1} P_{\lambda,\omega}\phi\|$ in* (3.1) *by*

$$\Theta_a(\phi) := \inf_{b \in \mathbb{Z}^2} \left\{ \langle b - a \rangle^\kappa \left\| T_b^{-1} P_{\lambda,\omega}\phi \right\| \right\} .$$

*Since $\Theta_a(\phi) \leq \left\| T_a^{-1} P_{\lambda,\omega}\phi \right\|$, this slightly improves* (3.3).

**Corollary 1.** *$H_\omega$ has pure point spectrum in the open set $\Xi_{\mathcal{I}}^{CL}$ for $\mathbb{P}$-a.e. $\omega$, with the corresponding eigenfunctions decaying faster than any sub-exponential. Moreover, we have (with $P_{\lambda,\omega} := P_{\{\lambda\},\omega}$)*

$$\mathbb{E}\left\{ \left\| \mu_\omega(\{\lambda\})\,(\operatorname{tr} P_{\lambda,\omega}) \right\|_{\mathrm{L}^\infty(I, \mathrm{d}\mu_\omega(\lambda))} \right\} \leq C_I < \infty, \tag{3.4}$$

*and hence for $\mathbb{P}$-a.e. $\omega$ the eigenvalues of $H_\omega$ in $\Xi_{\mathcal{I}}^{CL}$ are of finite multiplicity.*

It is well known that $H_\omega$ has pure point spectrum in $\Xi_{\mathcal{I}}^{\mathrm{CL}}$ with exponentially decaying eigenfunctions. Our point is that pure point spectrum follows directly from (3.3), also yielding sub-exponentially decaying eigenfunctions. The estimate (3.4) is new, and it immediately implies that for $\mathbb{P}$-a.e. $\omega$ the random operator $H_\omega$ has only eigenvalues with finite multiplicity in $\Xi_{\mathcal{I}}^{\mathrm{CL}}$.

If $H_\omega$ has pure point spectrum we might as well work with eigenfunctions, not generalized eigenfunctions. Given $\lambda \in \mathbb{R}$ and $a \in \mathbb{Z}^d$ we set

$$W_{\lambda,\omega}(a) := \begin{cases} \sup\limits_{\substack{\phi\in\mathcal{H}\\ P_{\lambda,\omega}\phi\neq 0}} \dfrac{\|\chi_a P_{\lambda,\omega}\phi\|}{\|T_a^{-1} P_{\lambda,\omega}\phi\|} & \text{if } P_{\lambda,\omega}\neq 0,\\ 0 & \text{otherwise,}\end{cases} \tag{3.5}$$

and

$$Z_{\lambda,\omega}(a) := \begin{cases} \dfrac{\|\chi_a P_{\lambda,\omega}\|_2}{\|T_a^{-1} P_{\lambda,\omega}\|_2} & \text{if } P_{\lambda,\omega}\neq 0,\\ 0 & \text{otherwise.}\end{cases} \tag{3.6}$$

$W_{\lambda,\omega}(a)$ and $Z_{\lambda,\omega}(a)$ are measurable functions of $(\lambda,\omega)$ for each $a \in \mathbb{Z}^d$. They are covariant, that is,

$$Y_{\lambda,\omega}(a) = Y_{\lambda,\tau(b)\omega}(a+b) \quad \text{for all } b\in\mathbb{Z}^d, \text{with } Y = W \text{ or } Y = Z\,. \tag{3.7}$$

It follows from (2.7) that $i_- P_{\lambda,\omega\,+} = \mathbf{P}_\omega(\lambda)\mu_\omega(\{\lambda\})$. Since $P_{\lambda,\omega}\neq 0$ if and only if $\mu_\omega(\{\lambda\})\neq 0$, we have $W_{\lambda,\omega}(a) = \mathbf{W}_{\lambda,\omega}(a)$ if $\mu_\omega(\{\lambda\})\neq 0$ and $W_{\lambda,\omega}(a) = 0$ otherwise. Combining this fact with the definition of the Hilbert-Schmidt norm and (3.2) we get

$$Z_{\lambda,\omega}(a) \le W_{\lambda,\omega}(a) \le \mathbf{W}_{\lambda,\omega}(a) \le \left(1+\tfrac{d}{4}\right)^{\frac{\kappa}{2}}. \tag{3.8}$$

**Remark 2.** $H_\omega$ has pure point spectrum in an open interval $I$ if and only if for $\mathbb{P}$-a.e. $\omega$ we have $W_{\lambda,\omega}(a) = \mathbf{W}_{\lambda,\omega}(a)$ for all $a\in\mathbb{Z}^d$ and $\mu_\omega$-a.e. $\lambda\in I$.

Thus we have the following corollary to Theorem 1.

**Corollary 2.** *Let $I$ be a bounded open interval with $\bar{I}\subset\mathcal{I}$. If $\bar{I}\subset\Xi^{CL}_{\mathcal{I}}$, $H_\omega$ has pure point spectrum in $\bar{I}$ for $\mathbb{P}$-a.e. $\omega$ and for all $\zeta\in]0,1[$ and $x,y\in\mathbb{Z}^d$ we have*

$$\mathbb{E}\left\{\left\|Z_{\lambda,\omega}(x)Z_{\lambda,\omega}(y)\right\|_{\mathrm{L}^\infty(I,\mathrm{d}\mu_\omega(\lambda))}\right\} \le C_{I,\zeta}\,\mathrm{e}^{-|x-y|^\zeta}, \tag{3.9}$$

$$\mathbb{E}\left\{\left\|W_{\lambda,\omega}(x)W_{\lambda,\omega}(y)\right\|_{\mathrm{L}^\infty(I,\mathrm{d}\mu_\omega(\lambda))}\right\} \le C_{I,\zeta}\,\mathrm{e}^{-|x-y|^\zeta}. \tag{3.10}$$

*Conversely, if $H_\omega$ has pure point spectrum in $\mathbb{P}$ for ı-a.e. $\omega$, and either* (3.9) *or* (3.10) *holds for some $\zeta\in]0,1[$, we have $I\subset\Xi^{\mathrm{CL}}_{\mathcal{I}}$.*

We now turn to almost sure consequences of Theorem 1.

**Corollary 3.** *Let $I$ be be a bounded open interval with $\bar{I}\subset\Xi^{CL}_{\mathcal{I}}$. The following holds for $\mathbb{P}$-a.e. $\omega$: $H_\omega$ has pure point spectrum in $I$ with finite multiplicity, so let $\{E_{n,\omega}\}_{n\in\mathbb{N}}$ be an enumeration of the (distinct) eigenvalues of $H_\omega$ in $I$, with $\nu_{n,\omega}$ being the (finite) multiplicity of the eigenvalue $E_{n,\omega}$. Then:*

(i) *Summable Uniform Decay of Eigenfunction Correlations (SUDEC): For each* $\zeta \in ]0,1[$ *and* $\varepsilon > 0$ *we have*

$$\|\chi_x \phi\| \|\chi_y \psi\| \le C_{I,\zeta,\varepsilon,\omega} \|T^{-1}\phi\| \|T^{-1}\psi\| \langle y \rangle^{d+\varepsilon} \, \mathrm{e}^{-|x-y|^\zeta}, \tag{3.11}$$

$$\|\chi_x \phi\| \|\chi_y \psi\| \le C_{I,\zeta,\varepsilon,\omega} \|T^{-1}\phi\| \|T^{-1}\psi\| \langle x \rangle^{\frac{d+\varepsilon}{2}} \langle y \rangle^{\frac{d+\varepsilon}{2}} \, \mathrm{e}^{-|x-y|^\zeta}, \tag{3.12}$$

*for all* $\phi, \psi \in \operatorname{Ran} P_{E_{n,\omega},\omega}$, $n \in \mathbb{N}$, *and* $x, y \in \mathbb{Z}^d$.

(ii) *Semi Uniformly Localized Eigenfunctions (SULE): There exist centers of localization* $\{y_{n,\omega}\}_{n\in\mathbb{N}}$ *for the eigenfunctions such that for each* $\zeta \in ]0,1[$ *and* $\varepsilon > 0$ *we have*

$$\|\chi_x \phi\| \le C_{I,\zeta,\varepsilon,\omega} \|T^{-1}\phi\| \langle y_{n,\omega} \rangle^{2(d+\varepsilon)} \, \mathrm{e}^{-|x-y_{n,\omega}|^\zeta}, \tag{3.13}$$

*for all* $\phi \in \operatorname{Ran} P_{E_{n,\omega},\omega}$, $n \in \mathbb{N}$, *and* $x \in \mathbb{Z}^d$. *Moreover, we have*

$$N_{L,\omega} := \sum_{n\in\mathbb{N};|y_{n,\omega}|\le L} \nu_{n,\omega} \le C_{I,\omega} L^d \quad \text{for all } L \ge 1. \tag{3.14}$$

(iii) *SUDEC and SULE for complete orthonormal sets: For each* $n \in \mathbb{N}$ *let* $\{\phi_{n,j,\omega}\}_{j\in\{1,2,\ldots,\nu_{n,\omega}\}}$ *be an orthonormal basis for the eigenspace* $\operatorname{Ran} P_{E_{n,\omega},\omega}$, *so* $\{\phi_{n,j,\omega}\}_{n\in\mathbb{N},j\in\{1,2,\ldots,\nu_{n,\omega}\}}$ *is a complete orthonormal set of eigenfunctions of* $H_\omega$ *with energy in* $I$. *Then for each* $\zeta \in ]0,1[$ *and* $\varepsilon > 0$ *we have*

$$\|\chi_x \phi_{n,i,\omega}\| \|\chi_y \phi_{n,j,\omega}\| \le C_{I,\zeta,\varepsilon,\omega} \sqrt{\alpha_{n,i,\omega}} \sqrt{\alpha_{n,j,\omega}} \langle y \rangle^{d+\varepsilon} \, \mathrm{e}^{-|x-y|^\zeta}, \tag{3.15}$$

$$\|\chi_x \phi_{n,i,\omega}\| \|\chi_y \phi_{n,j,\omega}\| \le C_{I,\zeta,\varepsilon,\omega} \sqrt{\alpha_{n,i,\omega}} \sqrt{\alpha_{n,j,\omega}} \langle x \rangle^{\frac{d+\varepsilon}{2}} \langle y \rangle^{\frac{d+\varepsilon}{2}} \, \mathrm{e}^{-|x-y|^\zeta}, \tag{3.16}$$

$$\|\chi_x \phi_{n,j,\omega}\| \le C_{I,\zeta,\varepsilon,\omega} \sqrt{\alpha_{n,j,\omega}} \langle y_{n,\omega} \rangle^{2(d+\varepsilon)} \, \mathrm{e}^{-|x-y_{n,\omega}|^\zeta}, \tag{3.17}$$

*for all* $n \in \mathbb{N}$, $i, j \in \{1,2,\ldots,\nu_{n,\omega}\}$, *and* $x, y \in \mathbb{Z}^d$, *where*

$$\alpha_{n,j,\omega} := \|T^{-1}\phi_{n,j,\omega}\|^2, \quad n \in \mathbb{N},\ j \in \{1,2,\ldots,\nu_{n,\omega}\}, \tag{3.18}$$

$$\sum_{j\in\{1,2,\ldots,\nu_{n,\omega}\}} \alpha_{n,j,\omega} = \mu_\omega(\{E_{n,\omega}\}) \quad \text{for all } n \in \mathbb{N}, \tag{3.19}$$

$$\sum_{n,\in\mathbb{N},j\in\{1,2,\ldots,\nu_{n,\omega}\}} \alpha_{n,j,\omega} = \sum_{n\in\mathbb{N}} \mu_\omega(\{E_{n,\omega}\}) = \mu_\omega(I). \tag{3.20}$$

**Remark 3.** The statements (i) and (ii) are essentially equivalent, and imply finite multiplicity for eigenvalues, while (iii) does not, see ref.[38]. Note that in (ii) eigenfunctions associated to the same eigenvalue have the same center of localization. It is easy to see that (3.11) implies (3.12), the reverse implication also being true up to a change in the constant – both forms of SUDEC are useful.

If $I$ is a bounded open interval with $\bar{I} \subset \Xi^{\mathrm{CL}}_{\mathcal{I}}$, it is known that that for $\mathbb{P}$-a.e. $\omega$ the operator $H_\omega$ has pure point spectrum in $I$ with exponentially decaying eigenfunctions.[23, 29, 41] The SULE property is also known with exponential decay. [32, 33] Combining the proof of ref.[31], Theorem 1.5, with the argument in refs.[23, 41] we also obtain SUDEC with exponential decay for $\mathbb{P}$-a.e. $\omega$.

**Theorem 2.** *Let $I$ be be a bounded open interval with $\bar{I} \subset \Xi_{\mathcal{I}}^{CL}$. For all $\phi \in \mathcal{H}_+$ and $\lambda \in I$ set $\alpha_{\lambda,\phi} := \|T^{-1}\mathbf{P}_\omega(\lambda)\phi\|^2$. The following holds for $\mathbb{P}$-a.e. $\omega$ and $\mu_\omega$-a.e. $\lambda \in I$: For all $\varepsilon > 0$ there exists $m_\varepsilon = m_{I,\varepsilon} > 0$ such that for all $\phi, \psi \in \mathcal{H}_+$ we have*

$$\begin{aligned} &\|\chi_x \mathbf{P}_\omega(\lambda)\phi\| \, \|\chi_y \mathbf{P}_\omega(\lambda)\psi\| \\ &\quad \leq C_{I,\varepsilon,\omega}\sqrt{\alpha_{\lambda,\phi}\alpha_{\lambda,\psi}}\, \mathrm{e}^{(\log\langle x\rangle)^{1+\varepsilon}} \mathrm{e}^{(\log\langle y\rangle)^{1+\varepsilon}} \mathrm{e}^{-m_\varepsilon|x-y|} \end{aligned} \tag{3.21}$$

*for all $x, y \in \mathbb{Z}^d$. In particular, it follows that $H_\omega$ has pure point spectrum in $I$ with exponentially decaying eigenfunctions.*

Unlike Theorem 1, Theorem 2 does not give a characterization of the region of complete localization. But it still implies that $H_\omega$ has only eigenvalues with finite multiplicity in $I$.[38]

Compared to the rather short and transparent proof of (3.12), the proof of (3.21) is quite technical and involved – an extra motivation for deriving (3.12).

We now turn to the characterization in terms of the decay of Fermi projections. We set $P_\omega^{(E)} := P_{]-\infty,E],\omega}$, the Fermi projection corresponding to the Fermi energy $E$.

**Theorem 3.** *Let $I$ and $I_1$ be bounded open intervals with $\bar{I} \subset I_1 \subset \bar{I}_1 \subset \Xi_{\mathcal{I}}^{CL}$. If $\bar{I} \subset \Xi_{\mathcal{I}}^{CL}$ Let $I$ be be a bounded open interval with $\bar{I} \subset \mathcal{I}$. If $\bar{I} \subset \Xi_{\mathcal{I}}^{CL}$, then for all $\zeta \in ]0,1[$ we have*

$$\mathbb{E}\left\{\sup_{E\in I} \left\|\chi_x P_\omega^{(E)} \chi_y\right\|_2^2\right\} \leq C_{I,\zeta}\, \mathrm{e}^{-|x-y|^\zeta} \quad \text{for all } x, y \in \mathbb{Z}^d. \tag{3.22}$$

*Conversely, if (3.22) holds for some $\zeta \in ]0,1[$, then $I \subset \Xi_{\mathcal{I}}^{CL}$.*

Again,the converse will still hold if we only have fast enough polynomial decay in (3.22). Its proof explicitly uses that slow enough transport (weaker than dynamical localization) implies that a multiscale analysis can be performed. The estimate (3.22) is known to hold for the Anderson model on the lattice with exponential decay, using the estimate given by the fractional moment method.[3]

## 4. SUMMABLE UNIFORM DECAY OF EIGENFUNCTION CORRELATIONS

In this section we prove Theorem 1 and its corollaries.

**Proof of Theorem 1:** Since $\bar{I} \subset \Xi_{\mathcal{I}}^{\mathrm{CL}}$, given any $\zeta$, $0 < \zeta < 1$, and $\alpha$, $1 < \alpha < \zeta^{-1}$, there is a length scale $L_0 \in 6\mathbb{N}$ and a mass $m > 0$, so if we set $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}}$, $k = 0, 1, \ldots,$ we have (2.8) for all $k = 0, 1, \ldots$, and $x, y \in \mathbb{Z}^d$ with $|x - y| > L_k + \varrho$.

Let $I \subset \Xi_{\mathcal{I}}^{\mathrm{CL}}$ be a bounded interval with $\bar{I} \subset \mathcal{I}$. Note that the quantity $\left\| \mathbf{W}_{\lambda,\omega}(x) \mathbf{W}_{\lambda,\omega}(y) \right\|_{\mathrm{L}^\infty(I, \mathrm{d}\mu_\omega(\lambda))}$ is measurable in $\omega$ since the $L^\infty$ norm on sets of finite measure is the limit of the $L^p$ norms as $p \to \infty$. (It is actually covariant in view of the way $\mathbf{P}_\omega(\lambda)$ is constructed (see ref.[42], Eq. (46)), and the fact that the measures $\mu_\omega$ and $\mu_{\tau(a)\omega}$ are equivalent.)

**Lemma 1.** *Let $\omega \in R(m, L, I, x, y)$ (defined in (2.9)). Then*

$$\left\| \mathbf{W}_{\lambda,\omega}(x) \mathbf{W}_{\lambda,\omega}(y) \right\|_{\mathrm{L}^\infty(I, \mathrm{d}\mu_\omega(\lambda))} \leq C_{I,m} \mathrm{e}^{-m \frac{L}{4}}. \tag{4.1}$$

**Proof:** Let $\omega \in R(m, L, I, x, y)$. Then for any $\lambda \in I$, either $\Lambda_L(x)$ or $\Lambda_L(y)$ is $(m, \lambda)$-regular for $H_\omega$, say $\Lambda_L(x)$. Given $\phi \in \mathcal{H}_+$, $\mathbf{P}_\omega(\lambda)\phi$ is a generalized eigenfunction of $H_\omega$ with eigenvalue $\lambda$ (perhaps the trivial eigenfunction 0), so it follows from the EDI[33], (2.15), using $\chi_x = \chi_{x,\frac{L}{3}} \chi_x$, that

$$\|\chi_x \mathbf{P}_\omega(\lambda)\phi\| \leq \tilde{\gamma}_I \|\Gamma_{x,L} R_{x,L}(\lambda) \chi_{x,L/3}\|_{x,L} \|\Gamma_{x,L} \mathbf{P}_\omega(\lambda)\phi\|. \tag{4.2}$$

Since $\Lambda_L(x)$ is $(m, \lambda)$-regular, we have that

$$\|\chi_x \mathbf{P}_\omega(\lambda)\phi\| \leq \tilde{\gamma}_I \mathrm{e}^{-m \frac{L}{2}} \|\Gamma_{x,L} \mathbf{P}_\omega(\lambda)\phi\| \leq C'_{I,m,d} \mathrm{e}^{-m \frac{L}{4}} \|T_x^{-1} \mathbf{P}_\omega(\lambda)\phi\|, \tag{4.3}$$

since

$$\|\Gamma_{x,L} \mathbf{P}_\omega(\lambda)\phi\| \leq C_d L^{d-1} \langle \tfrac{L+1}{2} \rangle^\kappa \|T_x^{-1} \mathbf{P}_\omega(\lambda)\phi\|. \tag{4.4}$$

Thus, using the bound (3.2) for the term in $y$, we get (4.1). □

If $\bar{I} \subset \Xi_{\mathcal{I}}^{\mathrm{CL}}$, given any $\zeta$, $0 < \zeta < 1$, and $\alpha$, $1 < \alpha < \zeta^{-1}$, there is a length scale $L_0 \in 6\mathbb{N}$ and a mass $m > 0$, so if we set $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}}$, $k = 0, 1, \ldots,$ we have (2.8) for all $k = 0, 1, \ldots$, and $x, y \in \mathbb{Z}^d$ with $|x - y| > L_k + \varrho$.

Thus given $x, y \in \mathbb{Z}^d$ and $k$ such that $L_{k+1} + \varrho \geq |x - y| > L_k + \varrho$, it follows from (4.1) that

$$\mathbb{E}\left\{ \left\| \mathbf{W}_{\lambda,\omega}(x) \mathbf{W}_{\lambda,\omega}(y) \right\|_{\mathrm{L}^\infty(I, \mathrm{d}\mu_\omega(\lambda))} ; R(m, L_k, I, x, y) \right\} \leq C_{I,m} \mathrm{e}^{-m \frac{L_k}{4}}. \tag{4.5}$$

On the complementary set we use the bound (3.2) for both terms, obtaining

$$\begin{aligned} &\mathbb{E}\left\{ \left\| \mathbf{W}_{\lambda,\omega}(x) \mathbf{W}_{\lambda,\omega}(y) \right\|_{\mathrm{L}^\infty(I, \mathrm{d}\mu_\omega(\lambda))} ; \omega \notin R(m, L_k, I, x, y) \right\} \\ &\quad \leq C_d \, \mathbb{P}\{\omega \notin R(m, L_k, I, x, y)\} \leq C_d \, \mathrm{e}^{-L_k^\zeta}. \end{aligned} \tag{4.6}$$

Since $L_{k+1} + \varrho \geq |x - y| > L_k + \varrho$, the estimate (3.3) now follows with $\frac{\zeta}{\alpha}$ instead of $\zeta$. Since $\zeta \in ]0, 1[$ and $1 < \alpha < \zeta^{-1}$ are otherwise arbitrary, (3.3) holds with any $\zeta \in ]0, 1[$.

To prove the converse, we use the following lemma.

**Lemma 2.** *For $\mathbb{P}$-a.e. $\omega$ we have*

$$\left\| \chi_x \mathbf{P}_\omega(\lambda) \chi_y \right\|_2^2 \leq C_d \langle x \rangle^{2\kappa} \langle y \rangle^{2\kappa} \mathbf{W}_{\lambda,\omega}(x) \mathbf{W}_{\lambda,\omega}(y) \tag{4.7}$$

*for all $x, y \in \mathbb{Z}^d$, $\lambda \in \mathbb{R}$.*

**Proof:** Let $\{\psi_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. We have

$$\begin{aligned} \left\| \chi_x \mathbf{P}_\omega(\lambda) \chi_y \right\|_2^2 &= \sum_{n \in \mathbb{N}} \left\| \chi_x \mathbf{P}_\omega(\lambda) \chi_y \psi_n \right\|^2 \\ &\leq [\mathbf{W}_{\lambda,\omega}(x)]^2 \sum_{n \in \mathbb{N}} \left\| T_x^{-1} \mathbf{P}_\omega(\lambda) \chi_y \psi_n \right\|^2 \\ &= [\mathbf{W}_{\lambda,\omega}(x)]^2 \left\| T_x^{-1} \mathbf{P}_\omega(\lambda) \chi_y \right\|_2^2 \\ &\leq C_d \langle x \rangle^{2\kappa} \langle y \rangle^{2\kappa} [\mathbf{W}_{\lambda,\omega}(x)]^2, \end{aligned} \tag{4.8}$$

where we used (2.6) and (2.2).

Since $\left\| \chi_x \mathbf{P}_\omega(\lambda) \chi_y \right\|_2 = \left\| \chi_y \mathbf{P}_\omega(\lambda) \chi_x \right\|_2$, the lemma follows. □

So now assume (3.3) holds for some $\zeta \in ]0, 1[$. By $\mathcal{B}_1 = \mathcal{B}_1(\mathbb{R})$ we denote the collection of real-valued Borel functions $f$ of a real variable with $\sup_{t \in \mathbb{R}} |f(t)| \leq 1$. Using the generalized eigenfunction expansion (2.7), Lemma 2, and (2.4), we get

$$\begin{aligned} \sup_{f \in \mathcal{B}_1} \| \chi_x f(H_\omega) P_\omega(I) \chi_0 \|_2 &\leq \sup_{f \in \mathcal{B}_1} \int_I |f(\lambda)| \, \| \chi_x \mathbf{P}_\omega(\lambda) \chi_0 \|_2 \, \mathrm{d}\mu_\omega(\lambda) \\ &\leq \int_I \| \chi_x \mathbf{P}_\omega(\lambda) \chi_0 \|_2 \, \mathrm{d}\mu_\omega(\lambda) \\ &\leq C_d^{\frac{1}{2}} \langle x \rangle^\kappa K_I \left\| \mathbf{W}_{\lambda,\omega}(x) \mathbf{W}_{\lambda,\omega}(0) \right\|_{\mathrm{L}^\infty(I, \mathrm{d}\mu_\omega(\lambda))}^{\frac{1}{2}}. \end{aligned} \tag{4.9}$$

Thus it follows from (3.3) that

$$\mathbb{E} \left\{ \sup_{f \in \mathcal{B}_1} \| \chi_x f(H_\omega) P_\omega(I) \chi_0 \|_2^2 \right\} \leq C_d C_{I,\zeta} K_I^2 \langle x \rangle^{2\kappa} \, \mathrm{e}^{-|x|^\zeta} \leq C'_{I,\zeta} \, \mathrm{e}^{-\frac{1}{2}|x|^\zeta}, \tag{4.10}$$

and hence for all $x, y \in \mathbb{Z}^d$ we have

$$\mathbb{E}\left\{\sup_{f\in\mathcal{B}_1} \left\|\chi_x f(H_\omega)P_\omega(I)\chi_y\right\|_2^2\right\} = \mathbb{E}\left\{\sup_{f\in\mathcal{B}_1} \left\|\chi_{x-y} f(H_\omega)P_\omega(I)\chi_0\right\|_2^2\right\}$$
$$\le C'_{I,\zeta}\, \mathrm{e}^{-\frac{1}{2}|x-y|^\zeta}. \tag{4.11}$$

It now follows from ref.[37], Theorem 4.2 that $I \subset \Xi_{\mathcal{I}}^{\mathrm{CL}}$ □

**Proof of Corollary 1:** Let us consider a bounded interval $I$ with $\bar{I} \subset \Xi_{\mathcal{I}}^{\mathrm{CL}}$. It follows from (4.16) that for any $\phi \in \mathcal{H}_+$ and $\mu_\omega$-a.e. $\lambda \in I$ we have

$$\|\chi_x \mathbf{P}_\omega(\lambda)\phi\|\,\|\chi_y \mathbf{P}_\omega(\lambda)\phi\| \le 2^\kappa C_{I,\xi,\omega}\mathrm{e}^{-|x-y|^\xi}\langle x\rangle^{3\kappa}\langle y\rangle^\kappa \|\phi\|_+^2$$
$$\le C_{I,\xi,d,\omega}\langle x\rangle^{3\kappa}\mathrm{e}^{-\frac{1}{2}|x-y|^\xi}\|\phi\|_+^2 \tag{4.12}$$

for all $x, y \in \mathbb{Z}^d$, where we used a consequence of (2.2), namely

$$\|T_a^{-1}\mathbf{P}_\omega(\lambda)\phi\| \le 2^{\frac{\kappa}{2}}\langle a\rangle^\kappa \|\mathbf{P}_\omega(\lambda)\phi\|_- \le 2^{\frac{\kappa}{2}}\langle a\rangle^\kappa\|\phi\|_+\,. \tag{4.13}$$

In particular, if $\mathbf{P}_\omega(\lambda)\phi \neq 0$ we can pick $x_0 \in \mathbb{Z}^d$ such that $\chi_{x_0}\mathbf{P}_\omega(\lambda)\phi \neq 0$, and thus

$$\|\chi_y \mathbf{P}_\omega(\lambda)\phi\| \le C_{I,\xi,d,\omega}\|\chi_{x_0}\mathbf{P}_\omega(\lambda)\phi\|^{-1}\|\phi\|_+^2\langle x_0\rangle^{3\kappa}\mathrm{e}^{-\frac{1}{2}|y-x_0|^\xi} \text{ for all } y \in \mathbb{Z}^d. \tag{4.14}$$

It follows that $\mathbf{P}_\omega(\lambda)\phi \in \mathcal{H}$, and hence $\mu_\omega$-a.e. $\lambda \in I$ is an eigenvalue of $H_\omega$. Thus $H_\omega$ has pure point spectrum in $I$, with the corresponding eigenfunctions decaying faster than any sub-exponential by (4.14). (See, e.g., ref.[42].)

In fact, these eigenvalues have finite multiplicity, a consequence of the estimate (3.4), which is proved as follows: Using (2.5) and (3.8), we have

$$\begin{aligned}
\mu_\omega(\{\lambda\})\,(\operatorname{tr} P_{\lambda,\omega}) &= \left\|T^{-1}P_{\lambda,\omega}\right\|_2^2(\operatorname{tr} P_{\lambda,\omega})\\
&\le C_d \sum_{x,y\in\mathbb{Z}^d}\langle x\rangle^{-2\kappa}\left\|\chi_x P_{\lambda,\omega}\right\|_2^2\left\|\chi_y P_{\lambda,\omega}\right\|_2^2\\
&\le C_d K_I^2 \sum_{x,y\in\mathbb{Z}^d}\langle x\rangle^{-2\kappa}\left(Z_{\lambda,\omega}(x)Z_{\lambda,\omega}(y)\right)^2\\
&\le C'_d K_I^2 \sum_{x,y\in\mathbb{Z}^d}\langle x\rangle^{-2\kappa}Z_{\lambda,\omega}(x)Z_{\lambda,\omega}(y),
\end{aligned} \tag{4.15}$$

and hence (3.4) follows from Remark 2 and (3.8) (or from (3.9)). □

**Lemma 3.** *Let $I$ be a bounded interval with $\bar{I} \subset \Xi_{\mathcal{I}}^{CL}$. Then for all $\xi \in ]0,1[$, $p \geq 1$, and $\mathbb{P}$-a.e. $\omega$ we have*

$$\left\| \sum_{x,y\in\mathbb{Z}^d} \mathrm{e}^{|x-y|^\xi} \langle x\rangle^{-2\kappa} \left[\mathbf{W}_{\lambda,\omega}(x)\mathbf{W}_{\lambda,\omega}(y)\right]^p \right\|_{\mathrm{L}^\infty(I,\mathrm{d}\mu_\omega(\lambda))} \leq C_{I,\xi,p,\omega} < \infty. \tag{4.16}$$

**Proof:** It follows from (3.3) and (3.2) that for any $\xi \in ]0,1[$ and $p \geq 1$ we have

$$\mathbb{E}\left\{ \sum_{x,y\in\mathbb{Z}^d} \mathrm{e}^{|x-y|^\xi} \langle x\rangle^{-2\kappa} \left\|\mathbf{W}_{\lambda,\omega}(x)\mathbf{W}_{\lambda,\omega}(y)\right\|^p_{\mathrm{L}^\infty(I,\mathrm{d}\mu_\omega(\lambda))} \right\} \leq C_{I,\xi,p} < \infty, \tag{4.17}$$

and hence (4.16) follows. □

In fact Lemma 3 holds for any $p > 0$ by modifying the proof of Theorem 1.

**Proof of Corollary 2:** Since when $H_\omega$ has pure point spectrum in $I$ for $\mathbb{P}$-a.e. $\omega$ the estimate (3.10) is the same as (3.3), the corollary with (3.10) follows immediately from Theorem 1. The estimate (3.9) follows immediately from from (3.10) in view of (3.8). To prove the converse from (3.9), note that if $\mu_\omega(\{\lambda\}) \neq 0$, we have, using (2.2) and (2.6),

$$\begin{aligned} \left\|\chi_x \mathbf{P}_\omega(\lambda)\chi_y\right\|_1 &= \mu_\omega(\{\lambda\})^{-1} \left\|\chi_x P_{\lambda,\omega}\chi_y\right\|_1 \\ &\leq \mu_\omega(\{\lambda\})^{-1} \left\|\chi_x P_{\lambda,\omega}\right\|_2 \left\|\chi_y P_{\lambda,\omega}\right\|_2 \\ &= \mu_\omega(\{\lambda\})^{-1} \left\|T_x^{-1} P_{\lambda,\omega}\right\|_2 \left\|T_y^{-1} P_{\lambda,\omega}\right\|_2 Z_{\lambda,\omega}(x) Z_{\lambda,\omega}(y) \\ &\leq C_d' \langle x\rangle^\kappa \langle y\rangle^\kappa Z_{\lambda,\omega}(x) Z_{\lambda,\omega}(y). \end{aligned} \tag{4.18}$$

Thus, if $H_\omega$ has pure point spectrum in $I$, (4.11) follows from (3.9), and hence $I \subset \Xi_{\mathcal{I}}^{\mathrm{CL}}$ by[37] Theorem 4.2. □

**Proof of Corollary 3:** Pure point spectrum almost surely in $I$ with eigenvalues of finite multiplicity follows from Corollary 1. It follows from Lemma 3 that for all $\xi \in ]0,1[$, $p \geq 1$, $x,y \in \mathbb{Z}^d$, $\phi,\psi \in \operatorname{Ran} P_{E_{n,\omega},\omega}$, $n \in N$, and $i,j \in \{1,2,\ldots,\nu_{n,\omega}\}$ we have

$$\begin{aligned} \|\chi_x\phi\|\|\chi_y\psi\| &\leq \left[W_{E_{n,\omega},\omega}(x) W_{E_{n,\omega},\omega}(y)\right] \left[\|T_x^{-1}\phi\|\|T_y^{-1}\psi\|\right] \\ &\leq 2^\kappa \langle x\rangle^\kappa \langle y\rangle^\kappa \|T_x^{-1}\phi\|\|T_y^{-1}\psi\| \left[C_{I,\xi,p,\omega}\langle y\rangle^{2\kappa} \mathrm{e}^{-|x-y|^\xi}\right]^{\frac{1}{p}} \\ &\leq C'_{I,\xi,p,\omega} \|T_x^{-1}\phi\|\|T_y^{-1}\psi\| \langle y\rangle^{\frac{2(p+1)\kappa}{p}} \mathrm{e}^{-\frac{1}{2p}|x-y|^\xi}, \end{aligned} \tag{4.19}$$

where we used (2.2).

The SUDEC estimate (3.11) for given $\varepsilon > 0$ and $\zeta \in ]0, 1[$ follows from (4.19) by working with $\frac{d}{2} < \kappa < \frac{d+\varepsilon}{2}$, choosing $p \geq 1$ such that $d + \varepsilon = \frac{2(p+1)\kappa}{p}$, and taking $\xi = \frac{1+\zeta}{2}$.

To prove the SULE-like estimate (3.13), for each $n \in \mathbb{N}$ we take a nonzero eigenfunction $\psi \in \operatorname{Ran} P_{E_{n,\omega},\omega}$, and pick $y_{n,\omega} \in \mathbb{Z}^d$ (not unique) such that

$$\|\chi_{y_{n,\omega}}\psi\| = \max_{y\in\mathbb{Z}^d} \|\chi_y \psi\|. \tag{4.20}$$

Since for all $a \in \mathbb{Z}^d$ and $\phi \in \mathcal{H}$ we have

$$\begin{aligned} \|T_a^{-1}\phi\|^2 &= \sum_{y\in\mathbb{Z}^d} \|\chi_y T_a^{-1}\phi\|^2 \leq \max_{y\in\mathbb{Z}^d} \|\chi_y\phi\|^2 \sum_{y\in\mathbb{Z}^d} \|\chi_y T_a^{-1}\|^2 \\ &= \max_{y\in\mathbb{Z}^d} \|\chi_y\phi\|^2 \sum_{y\in\mathbb{Z}^d} \|\chi_y T^{-1}\|^2 \leq C_d^2 \max_{y\in\mathbb{Z}^d} \|\chi_y\phi\|^2, \end{aligned} \tag{4.21}$$

we get

$$\|T_a^{-1}\psi\| \leq C_d \|\chi_{y_{n,\omega}}\psi\| \quad \text{for all } a \in \mathbb{Z}^d. \tag{4.22}$$

It now follows from (4.19), taking $\psi$ as in (4.20), $y = y_{n,\omega}$, using (4.22), and choosing $p$ and $\xi$ as above, that for all $x \in \mathbb{Z}^d$, $\psi \in \operatorname{Ran} P_{E_{n,\omega},\omega}$, and $i \in \{1, 2, \ldots, \nu_{n,\omega}\}$ we have

$$\|\chi_x\phi\| \leq C_d^{-1} C''_{I,\zeta,\varepsilon,\omega} \|T^{-1}\phi\| \langle y_{n,\omega}\rangle^{d+\varepsilon}\, \mathrm{e}^{-|x-y_{n,\omega}|^\zeta}, \tag{4.23}$$

which is just (3.13).

SUDEC and SULE for the complete orthonormal set $\{\phi_{n,j,\omega}\}_{n\in\mathbb{N},j\in\{1,2,\ldots,\nu_{n,\omega}\}}$ of eigenfunctions of $H_\omega$ with energy in $I$ follows. Note that the equalities (3.19) and (3.20) follow immediately from (2.3).

To prove (3.14), note that it follows from (3.17) that

$$\begin{aligned} &\left\| \chi_{\{|x-y_{n,\omega}|R\}}\phi_{n,j,\omega} \right\|^2 \\ &\quad \leq C^2_{I,\zeta,\varepsilon,\omega} \langle y_{n,\omega}\rangle^{2(d+\varepsilon)} \alpha_{n,j,\omega} \sum_{x\in\mathbb{Z}^d,\, |x-y_{n,\omega}|R} \mathrm{e}^{-|x-y_{n,\omega}|^\zeta} \\ &\quad \leq C'_{I,\zeta,\varepsilon,\omega} \langle y_{n,\omega}\rangle^{2(d+\varepsilon)} \alpha_{n,j,\omega} \mathrm{e}^{-\frac{1}{2}R^\zeta} \leq \tfrac{1}{2}, \end{aligned} \tag{4.24}$$

if we take

$$R = R_{n,j,\omega} \geq 2\left\{\log\left(2C'_{I,\zeta,\varepsilon,\omega}\langle y_{n,\omega}\rangle^{2(d+\varepsilon)}\alpha_{n,j,\omega}\right)\right\}^{\frac{1}{\zeta}}. \tag{4.25}$$

Given $L \geq 1$, we set

$$\begin{aligned} R_{L,\omega} &= 2\left\{\log\left(2C'_{I,\zeta,\varepsilon,\omega}\langle L\rangle^{2(d+\varepsilon)}\alpha_{n,j,\omega}\right)\right\}^{\frac{1}{\zeta}} \leq C''_{I,\zeta,\varepsilon,\omega}(\log L)^{\frac{1}{\zeta}}, \\ S_{L,\omega} &= L + 2R_{L,\omega} \leq C'''_{I,\zeta,\varepsilon,\omega} L. \end{aligned} \tag{4.26}$$

Note that if $|y_{n,\omega}| \leq L$ we have $\left\| \chi_{0,S_{L,\omega}} \phi_{n,j,\omega} \right\|^2 \frac{1}{2}$ for all $j \in \{1, 2, \ldots, \nu_{n,\omega}\}$. Thus, using (2.1) and (2.5), we get

$$\begin{aligned} \tfrac{1}{2} N_L &\leq \sum_{n \in \mathbb{N}, j \in \{1,2,\ldots,\nu_{n,\omega}\}} \|\chi_{0,S_{L,\omega}} \phi_{n,j,\omega}\|^2 = \|\chi_{0,S_{L,\omega}} P_{I,\omega}\|_2^2 \\ &\leq \sum_{a \in \mathbb{Z}^d \cap \Lambda_{S_{L,\omega}}(0)} \|\chi_a P_{I,\omega}\|_2^2 = \sum_{a \in \mathbb{Z}^d \cap \Lambda_{S_{L,\omega}}(0)} \|\chi_0 P_{I,\tau(-a)\omega}\|_2^2 \qquad (4.27) \\ &\leq C_d \sum_{a \in \mathbb{Z}^d \cap \Lambda_{S_{L,\omega}}(0)} \mu_{\tau(-a)\omega}(I) \leq C_d' S_{L,\omega}^d K_I \leq \tilde{C}_{I,\zeta,\varepsilon,\omega} K_I L^d, \end{aligned}$$

which yields (3.14). □

## 5. SUDEC WITH EXPONENTIAL DECAY

In this section we prove Theorem 2.

**Proof of Theorem 2:** Let us fix $\varepsilon > 0$. Since $\bar{I} \subset \Xi_{\mathcal{I}}^{\mathrm{CL}}$, we can pick $\zeta \in ]0, 1[$ and $\alpha \in ]1, \zeta^{-1}[$ and such that $\alpha < (1+\varepsilon)\zeta$ and there is a length scale $L_0 \in 6\mathbb{N}$ and a mass $m = m_\zeta > 0$, so if we set $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}}$, $k = 0, 1, \ldots,$ we have (2.8) for all $k = 0, 1, \ldots$, and $x, y \in \mathbb{Z}^d$ with $|x - y| > L_k + \varrho$. We fix $\rho \in ]\frac{2}{3}, 1[$ and $b > \frac{1+2\rho}{1-2\rho} > 1$. As in ref.[41] Proof of Theorem 6.4, we pick $\rho \in ]\frac{1}{3}, \frac{1}{2}[$ and $b > \frac{1+2\rho}{1-2\rho} > 1$, and for each $x_0 \in \mathbb{Z}^d$ and $k = 0, 1, \cdots$ define the discrete annuli

$$A_{k+1}(x_0) = \left\{ \Lambda_{2bL_{k+1}}(x_0) \setminus \Lambda_{2L_k}(x_0) \right\} \cap \mathbb{Z}^d, \tag{5.1}$$

$$\tilde{A}_{k+1}(x_0) = \left\{ \Lambda_{\frac{2b}{1+\rho} L_{k+1}}(x_0) \setminus \Lambda_{\frac{2}{1-\rho} L_k}(x_0) \right\} \cap \mathbb{Z}^d. \tag{5.2}$$

We consider the event

$$F_k = \bigcap_{y \in \mathbb{Z}^d,\, \log\langle y\rangle \leq (mL_{k+1})^{(1+\varepsilon)^{-1}}} \ \bigcap_{x \in A_{k+1}(y)} R(m, L_k, I, x, y), \tag{5.3}$$

with $R(m, L, I, x, y)$ given in (2.9). It follows from (2.8) that $\sum_{k=1}^\infty \mathbb{P}(F_k^c) < \infty$, so that the Borel-Cantelli Lemma applies and yields an almost-surely finite $k_1(\omega)$, such that for all $k > k_1(\omega)$, if $E \in I$ and $\log\langle y\rangle \leq (mL_{k+1})^{(1+\varepsilon)^{-1}}$, either $\Lambda_{L_k}(y)$ is $(\omega, m, E)$-regular or $\Lambda_{L_k}(x)$ is $(\omega, m, E)$-regular for all $x \in A_k(y)$. For convenience we require $k_1(\omega) > 1$.

Using (ref.[41], Lemma 6.2) we conclude that for all $y \in \mathbb{Z}^d$, $\mathbb{P}$-a.e. $\omega$, and $\mu_\omega$-a.e. $\lambda \in \mathcal{I}$, there exists a finite $k_2 = k_2(y, \omega, \lambda)$ such that for all $k > k_2$ we have that $\Lambda_{L_k}(y)$ is $(\omega, m, \lambda)$-singular, and moreover $\Lambda_{L_{k_2}}(y)$ is $(\omega, m, \lambda)$-regular unless $k_2(\omega, y, \lambda) = 0$.

For each $y \in \mathbb{Z}^d$ we define $k_3 := k_3(y)$ by

$$\left(mL_{k_3}\right)^{(1+\varepsilon)^{-1}} < \log\langle y\rangle \le \left(mL_{k_3+1}\right)^{(1+\varepsilon)^{-1}}, \tag{5.4}$$

when possible, with $k_3(y) = -1$ otherwise.

We now set

$$k_* := k_*(\omega, y, \lambda) = \max\{k_1(\omega), k_3(y), k_2(\omega, y, \lambda) + 1\}; \tag{5.5}$$

note that $1 \le k_*(\omega, y, \lambda) < \infty$ for $\mathbb{P}$-a.e. $\omega$, and $\mu_\omega$-a.e. $\lambda \in \mathcal{I}$.

Let $\phi, \psi \in \mathcal{H}_+$ be given. Then for $\mathbb{P}$-a.e. $\omega$, and $\mu_\omega$-a.e. $\lambda \in \mathcal{I}$, if $k \ge k_*$ the box $\Lambda_{L_k}(y)$ is $(\omega, m, \lambda)$-singular and thus $\Lambda_{L_k}(x)$ is $(\omega, m, \lambda)$-regular for all $x \in A_{k+1}(y)$. It follows, as in (ref.[4], Proof of Theorem 6.4), that for all $x \in \tilde{A}_{k+1}(y)$ we have

$$\|\chi_x \mathbf{P}_\omega(\lambda)\psi\| \le C_{d,m}\langle y\rangle^\kappa \|T^{-1}\mathbf{P}_\omega(\lambda)\psi\| \mathrm{e}^{-m_\rho|x-y|}, \tag{5.6}$$

where $m_\rho = \frac{\rho(3\rho-1)}{2} m \in ]0, m[$. It remains to consider the case when $x \in \Lambda_{\frac{2}{1-\rho}L_{k_*}}(y) \cap \mathbb{Z}^d$. If $k_* = \max\{k_1(\omega), k_3(y)\} > k_2(\omega, y, \lambda)$, we use (3.2) and, if $k_* = k_3(y)$, (5.4), getting

$$\|\chi_x \mathbf{P}_\omega(\lambda)\psi\| \le C_d \|T_x^{-1}\mathbf{P}_\omega(\lambda)\psi\| \mathrm{e}^{mL_{k_*}} \mathrm{e}^{-mL_{k_*}} \tag{5.7}$$
$$\le \begin{cases} C_d\langle x\rangle^\kappa \|T^{-1}\mathbf{P}_\omega(\lambda)\psi\| \mathrm{e}^{(\log\langle y\rangle)^{1+\varepsilon}} \mathrm{e}^{-m|x-y|} & \text{if } k_* = k_3(y) \\ C_d\langle x\rangle^\kappa \|T^{-1}\mathbf{P}_\omega(\lambda)\psi\| \mathrm{e}^{mL_{k_1(\omega)}} \mathrm{e}^{-m|x-y|} & \text{if } k_* = k_1(\omega) \end{cases}$$

Estimating $\|\chi_y \mathbf{P}_\omega(\lambda)\phi\|$ by (3.2), we get the bound

$$\|\chi_x \mathbf{P}_\omega(\lambda)\psi\| \|\chi_y \mathbf{P}_\omega(\lambda)\phi\|$$
$$\le C_{d,\omega}\langle x\rangle^\kappa \langle y\rangle^{2\kappa} \sqrt{\alpha_{\lambda,\phi}\alpha_{\lambda,\psi}}\, \mathrm{e}^{(\log\langle y\rangle)^{1+\varepsilon}} \mathrm{e}^{-m'|x-y|}, \tag{5.8}$$

with $m' = m_\rho$. If $k_* = k_2(\omega, y, \lambda) + 1 > \max\{k_1(\omega), k_3(y)\}$, we must have $k_2 \ge 1$ and hence $\Lambda_{L_{k_2}}(y)$ is $(\omega, m, \lambda)$-regular. Using (4.3) and (2.2), we get

$$\|\chi_y \mathbf{P}_\omega(\lambda)\phi\| \le C_{d,I,m}\langle y\rangle^\kappa \|T^{-1}\mathbf{P}_\omega(\lambda)\phi\| \mathrm{e}^{-m\frac{L_{k_2}}{4}}. \tag{5.9}$$

If $x \in \Lambda_{\frac{2}{1-2\rho}L_{k_2}}(y) \cap \mathbb{Z}^d$, we may bound the term in $x$ by (3.2) and get (5.8) with $m' = \frac{(1-2\rho)m}{4}$ and another constant $C_{d,\omega}$. Since $x \in \Lambda_{\frac{2}{1-\rho}L_{k_2+1}}(y) \cap \mathbb{Z}^d$, we cannot have $x \notin \Lambda_{\frac{2b}{1+2\rho}L_{k_2+1}}(y) \cap \mathbb{Z}^d$ by our choice of $b$ and $\rho$. Thus the only remaining case is when $x \in \tilde{A}'_{k_2+1}(y)$, where $\tilde{A}'_{k_2+1}(y)$ is defined as in (5.2) but with $2\rho$ substituted for $\rho$. If all boxes $\Lambda_{L_{k_2}}(x')$ with $|x' - x| \le \rho|x - y|$ are $(\omega, m, \lambda)$-regular, the argument in (ref.[4], Proof of Theorem 6.4) still applies, and hence we also get (5.6) and (5.8) with with $m' = m_\rho$. If not, there exists $x' \in \tilde{A}_{k_2+1}(y)$ with $|x' - x| \le \rho|x - y|$ such that $\Lambda_{L_{k_2}}(x')$ is $(\omega, m, \lambda)$-singular. Clearly, $x' \in$

$\tilde{A}_{k_2+1}(y)$ if and only if $y \in \tilde{A}_{k_2+1}(x')$. In addition, since $k_3(y) \le k_2(\omega, y, \lambda)$ we have $k_3(x') \le k_2(\omega, y, \lambda) + 1$, as

$$\log \langle x' \rangle \le \tfrac{1}{2} \log 2 + \log \langle y \rangle + \log \langle bL_{k_2+1} \rangle \le \left(mL_{k_2+1}\right)^{(1+\varepsilon)^{-1}}. \tag{5.10}$$

Thus, as $k_2 \ge k_1(\omega)$, we can apply the argument leading to (5.6) in the annulus $\tilde{A}_{k_2+1}(x')$, obtaining

$$\|\chi_y \mathbf{P}_\omega(\lambda)\phi\| \le C_{d,m} \langle x' \rangle^\kappa \|T^{-1}\mathbf{P}_\omega(\lambda)\phi\| \mathrm{e}^{-m_\rho |x'-y|} \tag{5.11}$$

$$\le C'_{d,m} \langle y \rangle^\kappa \|T^{-1}\mathbf{P}_\omega(\lambda)\phi\| \mathrm{e}^{-\rho(1-\rho)m_\rho |x-y|}, \tag{5.12}$$

where we used $|x'-x| \le \rho|x-y|$ and $|x'-y||x-y| - |x'-x|(1-\rho)|x-y|$. Estimating $\|\chi_x \mathbf{P}_\omega(\lambda)\psi\|$ by (3.2), we get the bound

$$\|\chi_x \mathbf{P}_\omega(\lambda)\psi\| \|\chi_y \mathbf{P}_\omega(\lambda)\phi\| \le C_{d,\omega} \langle x \rangle^\kappa \langle y \rangle^\kappa \sqrt{\alpha_{\lambda,\phi}\alpha_{\lambda,\psi}}\, \mathrm{e}^{-m'|x-y|} \tag{5.13}$$

with $m' = \rho(1-\rho)m_\rho$.

The thorem is proved. □

## 6. DECAY OF THE FERMI PROJECTION

In this section we prove Theorem 3.

**Proof of Theorem 3:** Let $I$ and $I_1$ be bounded open intervals with $\bar{I} \subset I_1 \subset \bar{I}_1 \subset \Xi^{\mathrm{CL}}_{\mathcal{I}}$. It follows from (ref.[33], Theorem 3.8) that for all $\zeta \in ]0,1[$ we have

$$\mathbb{E}\left\{ \sup_{f \in \mathcal{B}_1} \left\| \chi_x f(H_\omega) P_\omega(I_1) \chi_y \right\|_2^2 \right\} \le C_{I_1,\zeta}\, \mathrm{e}^{-|x-y|^\zeta} \quad \text{for all } x, y \in \mathbb{Z}^d. \tag{6.1}$$

We write $I = (\alpha, \beta)$, and fix $\delta = \frac{1}{2}\mathrm{dist}(I, \partial I_1) > 0$. Given $\zeta \in ]0,1[$, we choose $\zeta' \in ]\zeta, 1[$. Since $H_\omega$ is semibounded, we can choose $\gamma > -\infty$ such that $\Sigma \subset ]\gamma, \infty[$. We pick a $\mathrm{L}^1$-Gevrey function $g$ of class $\frac{1}{\zeta'}$ on $]\gamma, \infty[$, such that $0 \le g \le 1$, $g \equiv 1$ on $]-\infty, \alpha-\delta]$ and $g \equiv 0$ on $]\beta+\delta, \infty[$. (See ref.[8], Definition 1.1); such a function always exists.) For all $E \in I$ we have $P^{(E)}_\omega = g(H_\omega) + f_E(H_\omega)$, where $f_E(t) = \chi_{]-\infty,E]}(t) - g(t) \in \mathcal{B}_1$, with $f_E(H_\omega) = f_E(H_\omega)P_\omega(I_1)$. Using (ref.[8], Theorem 1.4), for $\mathbb{P}$-a.e. $\omega$ we have

$$\left\| \chi_x g(H_\omega) \chi_y \right\| \le C_{g,\zeta,\zeta'}\, \mathrm{e}^{-C_{g,\zeta,\zeta'}|x-y|^\zeta} \quad \text{for all } x, y \in \mathbb{Z}^d. \tag{6.2}$$

On the other hand, it follows from (ref.[33] Eq. (2.36)) and the covariance (2.1) that for $\mathbb{P}$-a.e. $\omega$

$$\left\| \chi_x g(H_\omega) \chi_y \right\|_1 \le \|\chi_x g(H_\omega) \chi_x\|_1^{\frac{1}{2}} \left\| \chi_y g(H_\omega) \chi_y \right\|_1^{\frac{1}{2}} \le C_g \quad \text{for all } x, y \in \mathbb{Z}^d. \tag{6.3}$$

Since $\|A\|_2^2 \leq \|A\|\,\|A\|_1$ for any operator $A$, we get

$$\left\|\chi_x g(H_\omega)\chi_y\right\|_2^2 \leq C'_{g,\zeta,\zeta'}\,\mathrm{e}^{-C'_{g,\zeta,\zeta'}|x-y|^\zeta} \quad \text{for all } x, y \in \mathbb{Z}^d. \tag{6.4}$$

The estimate (3.22) for all $\zeta \in ]0, 1[$ now follows from (6.1) and (6.4).

To prove the converse, let us suppose (3.22) holds for some $\zeta \in ]0, 1[.)$ Let $\mathcal{X} \in C^\infty_{c,+}(I)$. By the spectral theorem,

$$\begin{aligned} \mathrm{e}^{-itH_\omega}\mathcal{X}(H_\omega) &= \int \mathrm{e}^{-itE}\mathcal{X}(E)P_\omega(\mathrm{d}E) = -\int \left(\mathrm{e}^{-itE}\mathcal{X}(E)\right)' P_\omega^{(E)}\mathrm{d}E \\ &= -\int_I \left(\mathrm{e}^{-itE}\mathcal{X}(E)\right)' P_\omega^{(E)}\mathrm{d}E. \end{aligned} \tag{6.5}$$

Thus for all $n > 0$ we have

$$\left\|\langle x\rangle^{\frac{n}{2}}\mathrm{e}^{-itH_\omega}\mathcal{X}(H_\omega)\chi_0\right\|_2 \leq C_{\mathcal{X}}(1+t)\int_I \left\|\langle x\rangle^{\frac{n}{2}}P_\omega^{(E)}\chi_0\right\|_2 \mathrm{d}E, \tag{6.6}$$

and hence

$$\begin{aligned} &\mathbb{E}\left\{\left\|\langle x\rangle^{\frac{n}{2}}\mathrm{e}^{-itH_\omega}\mathcal{X}(H_\omega)\chi_0\right\|_2^2\right\} \\ &\quad\leq C_{\mathcal{X}}^2(1+t)^2\mathbb{E}\left\{\left\{\int_I \left\|\langle x\rangle^{\frac{n}{2}}P_\omega^{(E)}\chi_0\right\|_2 \mathrm{d}E\right\}^2\right\} \\ &\quad\leq C_{\mathcal{X}}^2(1+t)^2|I|\int_I \mathbb{E}\left\{\left\|\langle x\rangle^{\frac{n}{2}}P_\omega^{(E)}\chi_0\right\|_2^2\right\}\mathrm{d}E \leq C_{\mathcal{X},I,n,\zeta}(1+t)^2, \end{aligned} \tag{6.7}$$

where we used (3.22) to get the last inequality. It follows that

$$\begin{aligned} \mathcal{M}(n,\mathcal{X},T) &:= \frac{2}{T}\int_0^\infty \mathrm{e}^{-\frac{2t}{T}}\mathbb{E}\left\{\left\|\langle x\rangle^{\frac{n}{2}}\mathrm{e}^{-itH_\omega}\mathcal{X}(H_\omega)\chi_0\right\|_2^2\right\}\mathrm{d}t \\ &\leq C'_{\mathcal{X},I,n,\zeta}(1+T^2), \end{aligned} \tag{6.8}$$

hence

$$\liminf_{T\to\infty}\frac{1}{T^\alpha}\mathcal{M}(n,\mathcal{X},T) < \infty \quad \text{for all } \alpha \geq 2 \text{ and } n > 0. \tag{6.9}$$

It now follows from (ref.[37], Theorem 2.11) that $I \subset \Xi^{\mathrm{CL}}_{\mathcal{I}}$. □

## ACKNOWLEDGMENTS

F.G. is currently visiting the Université de Paris Nord with support from the CNRS. A.K. was supported in part by NSF Grant DMS-0200710.

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